GROUPS WITH STRONGLY SELF-CENTRALIZING 3-CENTRALIZERS

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ABSTRACT

Let G be a finite group with a subgroup M which is the centralizer in G of each of its nonidentity elements and let 3 divide the order of M. Such groups are classified under the assumption that either $q = [N_G(M) \colon M] = 2^s$; $s \geqq 0$ and $s \neq 3$or $q < 8$.

1. Introduction

Let G be a finite group with a subgroup M such that M is the centralizer in G of each of its nonidentity elements. Assume also that 3 divides $|M|$. Then the groups G were classified by the author in [1] under the assumption that $q = [N_G(M) \colon M]$ is odd and Stewart** [3] has classified them under the assumption that $q \leqq 2$.

The purpose of this paper is to generalize Stewart's results in two directions. In Theorem 1 we consider the cases $q = 2^s$, $s \geqq 0$ and obtain a complete classification of groups involved, with the exception of $q = 8$. In particular, it is shown that $PSL(2,9)$ is the only simple group of the considered type with $q = 4$. In Theorem 2 a complete classification of groups with $q < 8$ is derived from Theorem 1 and the results of [1] .

The proofs of Theorem 1 and 2 rely heavily on [1]. Although in each case the results of [1] which were used were explicitly indicated, the reader is expected to be familiar with [1].

Our main result is

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THEOREM 1. *Let G be a finite group and suppose that G contains a subgroup M with the following properties*:

(i) *Whenever* $1 \neq x \in M$ *then* $C_G(x) = M$;

(ii) $3 \mid |M|$;

(iii) $q = [N_G(M): M] = 2^s$, *where s is a nonnegative integer.*

Then one of the following statements holds:

(a) *G has a normal nilpotent subgroup N such that* $G/N \cong N_G(M)$. *If* $q > 2$ *then* $|N| = 1$.

(b) $q = 8$, *the Sylow* 3-*subgroup P of M satisfies* $|\Omega_1(P)| = 9$.

(c) $q = 4$, $|M| = 9$, $G \cong PSL(2,9)$.

(d) $q = 2$, *M is cyclic and G has a normal elementary abelian* 2-*subgroup N such that* $G/N \cong PSL(2,2^k), k > 1$.

(e) $q = 2$, $G \cong PSL(2, r)$, *r odd.*

As a corollary we get

THEOREM 2. *Let G be a finite group and suppose that G contains a subgroup M with the following properties*:

(i) *Whenever* $1 \neq x \in M$ *then* $C_G(x) = M$;

(ii) $3 \mid |M|$;

(iii) $q = [N_G(M): M] < 8$.

Then one of the statements (a), (c), (d) *or* (e) *in Theorem* 1 *holds.*

2. Proof of Theorem 1.

Let G be a counter-example of minimal order. Let $|G| = g$, $|M| = m$ and $|N_G(M)| = qm$. Let, finally, P be the Sylow 3-subgroup of M and t_0 be the number of conjugate classes of elements of order 3 in G. We will proceed with a series of lemmas.

LEMMA 1. *G is simple and* $q = 4$, $t_0 = 2$, $m \geqq 27$.

PROOF. If $q \leqq 2$, then by Theorem A in [3] one of the conclusions (a), (d) or (e) holds and G is not a counter-example. Thus $q > 2$ and consequently P is not cyclic.

If $t_0 = 1$ then $2^s = q = 3^w - 1$, where $s > 1$ and $3^w = |\Omega_1(P)|$. Hence $s = 3$ $w = 2$ and G is of type (b), a contradiction. Thus $t_0 \geqq 2$ and in particular $q \neq m - 1$.

It follows from our assumptions and the above remarks that G and M satisfy Hypothesis B in [1]. Suppose that M is an elementary abelian 3-group of order 3^u. Then by Theorem 4.3 in [1] G is a simple group and $q = (m-1)/2$. This yields $2^{s+1} = 3^u - 1$ and consequently $s = u = 2$; thus $q = 4$ and $m = 9$. But then it follows from Theorem 13.3 in [2] that $G \cong PSL(2,9)$ and G is not a counter-example after all.

Thus M is not elementary abelian and by Theorem 2.2 in [1] $q < (m-1)^{\frac{1}{2}}$. Corollary 4.4 in [1] then yields $t_0 \mid q$, hence $t_0 = 2^v$, $v \geqq 1$. Let $|\Omega_1(P)| = 3^w$; then

$$2^{s+v} = qt_0 = 3^w - 1$$

and as $s, v \geqq 1$, $s = 2$ and $v = 1$. Consequently $q = 4$ and $t_0 = 2$.

Suppose that G is not simple. Since P s not cyclic, it follows from Corollary 2.2 in [1] that G contains a normal simple subgroup G^* of index 2. As $|N_{G^*}(M)| = 2m$, it follows by induction that M is cyclic, a contradiction. Thus G is simple.

Since P is noncyclic and M is not elementary abelian it follows that $m \geqq 27$.

LEMMA 2. *Let z be the degree of exceptional characters of G. Then $z \geqq 2m-4$.*

PROOF. As $q = 4 < m^{1/2} - 1$ and $t_0 = 2$, Corollaries 4.4 and 4.5 in [1] yield

$$g = \frac{2m^2}{-2(m-8)/z + S} \tag{1}$$

where S is defined by

$$S = 1 + \frac{\varepsilon_2}{u_2 m + \varepsilon_2} + \frac{\varepsilon_3}{u_3 m + \varepsilon_3} + \frac{\varepsilon_4}{u_4 m + \varepsilon_4}$$

where $\varepsilon_i = \pm 1$, $i = 2,3,4$, are the values taken on $M^{\#}$ by the nonprincipal nonexceptional characters θ_i, $i = 2,3,4$ of G, respectively ([1], Theorem 3.1, part I), and $u_i m + \varepsilon_i$, $i = 2,3,4$ are their degrees (see [1], Section 3). Since G is simple, $u_i \geqq 1$ for $i = 2,3,4$ and it follows that

$$(m+4)/(m+1) \geqq S \geqq (m-4)/(m-1). \tag{2}$$

Equations (1) and (2) yield

$$(m+4)/(m+1) \geqq S \geqq 2(m-8)/z$$

and consequently, in view of $m \geqq 27$

$$z \geqq 2(m-8)(m+1)/(m+4) > m-4.$$

Since by [1], Section 3 and Corollary 4.5, $z = am - q$, it follows from the above inequality that $z \geqq 2m-4$.

LEMMA 3. *Theorem 1 is true.*

PROOF. It follows from (1), (2) and Lemma 2 that

$$g \leqq \frac{2m^2}{-\frac{m-8}{m-2}+\frac{m-4}{m-1}}$$

$$= 2m(m-1)(m-2)/3.$$

However, G has $(m-1)/4$ exceptional characters of degree $2m-4$ at least and consequently $g > (m-1)(m-2)^2$. The two inequalities for g require $m < 6$, a contradiction. Thus G does not exist and Theorem 1 is true.

3. Proof of Theorem 2.

If $q = 2^s$, $s \geqq 0$, then by Theorem 1 one of the statements (a), (c), (d) or (e) holds. Let $|M| = m$; since $(q,m) = 1$, it remains to consider the values $q = 5$ and $q = 7$. If M is normal in G, then (a) holds. If $q = m-1$, then M is an elementary abelian 3-group, in contradiction to the fact that $q+1$ is not a power of 3. Thus we may assume that G and M satisfy Hypothesis B in [1] and consequently, by Corollary 4.6 and Theorem 4.4 in [1], M is elementary abelian and $q = (m-1)/2$. However this is again impossible since $m = 2q+1$ is not a power of 3 for $q = 5$ or 7. The proof of Theorem 2 is complete.

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